

NASA-TM-4042 19880016801

Accuracy of Analytic Energy Level Formulas Applied to Hadronic Spectroscopy of Heavy Mesons

**Forooz F. Badavi, John W. Norbury,
John W. Wilson, and Lawrence W. Townsend**

JULY 1988

LIBRARY COPY

JUL 13 1988

**LANGLEY RESEARCH CENTER
LIBRARY, NASA
HAMPTON, VIRGINIA**

NASA

NASA Technical Memorandum 4042

Accuracy of Analytic Energy Level Formulas Applied to Hadronic Spectroscopy of Heavy Mesons

Forooz F. Badavi
Planning Research Corporation
Hampton, Virginia

John W. Norbury
Washington State University
Pullman, Washington

John W. Wilson and Lawrence W. Townsend
Langley Research Center
Hampton, Virginia



National Aeronautics
and Space Administration

Scientific and Technical
Information Division

1988

Symbols

A	normalization constant, equation (A12)
$Ai[]$	Airy function, equation (A11)
c	speed of light, fm/s
E_n	matrix element of $q\bar{q}$ pair interaction, equation (1)
$f[], g[]$	terms in approximation to Airy function, equation (A28)
\hbar	Planck's constant, MeV-s
J	total meson spin
k	strength of potential, equations (2) and (7)
L	orbital meson momentum
$L_{n_r}[]$	associated Laguerre polynomial
ℓ	orbital quantum number
M_n	mass of the eigenstate, equation (1)
m	magnetic quantum number
$m_q, m_{\bar{q}}$	mass of quark or antiquark (q may be b , c , or t)
N, M	transformation variables, equations (B22)
n	radial quantum number
n_r	radial node number, equation (4)
$P[]$	associated Legendre polynomial
$R[]$	radial component of wave function, $\text{fm}^{-3/2}$
R_n, R_{n_r}	zeros of Airy function
r	radial distance, fm
S	intrinsic meson spin
$U[]$	reduced radial component of wave function, $\text{fm}^{-1/2}$
V_0	constant additive potential, equation (1)
$v(r)$	potential function, equations (2) and (7)
Y_ℓ^m	spherical harmonics
z	arbitrary function
$\Gamma[]$	gamma function
θ	zenith component of wave function
λ, ξ, η	transformation variables, equations (A6) and (A8)
μ	reduced mass
ϕ	azimuthal component of wave function
ψ	complete wave function
ω	angular frequency

Introduction

With the present interest in understanding quark confinement, a large amount of theoretical work based on nonrelativistic models (ref. 1), relativistic models (ref. 2), and field theoretic models (ref. 3) has been done to obtain particle mass spectra of mesons as bound states of quarks. According to the quark model, mesons are composed of a tightly bound quark-antiquark pair ($q\bar{q}$) which interacts with an external probe like a single collective object. Of particular interest are the nonrelativistic quark models, which have proven to be very successful for describing the spectroscopy, decay amplitudes, static properties, and binding forces of mesons and baryons (refs. 4 through 11). These nonrelativistic models are most successful when they are applied to systems where the energy level spacings are much smaller than the energy equivalent of the constituent masses (i.e., for systems which are, in fact, nonrelativistic) such as mesons consisting of the heavy charm, bottom, and top quarks. Through the use of nonrelativistic models, several independent lines of arguments suggest that the confinement of quarks may be based on an interaction which increases for large quark separation in proportion to the separation itself. This argument is assumed to be the dominating mechanism which explains the energy eigenvalues in the $q\bar{q}$ pair system for mesons.

In this paper, the picture that charm, bottom, and top q and \bar{q} are interacting with each other through a linear potential is used to analyze the particle spectrum of observed (charm, bottom) and predicted (top) mesons. Based on this picture, the quarks in the $q\bar{q}$ pair initially freely recede from each other in accordance with the ideas of asymptotic freedom; however, as the separation distance increases, a linear potential emerges and a bound state is formed. It is then assumed that, as long as the quarks are in an energy eigenstate, their lifetime is reasonably long and their eigenstates can be completely characterized by a linear potential.

Through the use of experimentally observed resonances at 3096 and 3686 MeV/ c^2 for the charmed meson, the two parameters of the linear potential k and V_0 (the strength of the interaction potential and an additive constant) are determined. Then, in the center-of-mass system of the bound $q\bar{q}$ pair, the nonrelativistic Schrödinger equation is exactly ($\ell = 0$) and asymptotically ($\ell \neq 0$) solved for a linear potential to obtain the remaining energy eigenvalues for charm and all eigenvalues of the bottom and top mesons. One then adds twice the quark mass plus a constant to these energy levels to obtain the meson

mass as

$$M_{nc}^2 = 2m_q c^2 + V_0 + E_n \quad (1)$$

where M_{nc}^2 is the meson mass, $m_q c^2$ is the constituent quark mass, V_0 is a constant additive term, and E_n is the matrix element of the $q\bar{q}$ pair interaction.

The quark confinement potential has been treated for many years as a harmonic potential mainly because of its closed-form solution in both the nonrelativistic and relativistic regimes (refs. 12 through 16). Here the exact nonrelativistic solutions for the eigenvalues and eigenfunctions of the harmonic potential are presented and the resulting energy levels are compared with those of the linear potential to validate the use of these two potential forms.

Potential Models

Linear Potential

The three-dimensional linear potential is defined as

$$v(r) = kr \quad (2)$$

where k is the constant describing the strength of the interaction potential and r is the $q\bar{q}$ pair separation. With \hbar in units of MeV-s and c in units of fm/s, the energy eigenvalues for an S state ($\ell = 0$) are obtained from the exact solution of the nonrelativistic Schrödinger equation as

$$E_{nr} = \left[\frac{k^2 (\hbar c)^2}{\mu c^2} \right]^{1/3} |R_n| \quad (3)$$

where

$$n_r = n - 1 \quad (n = 1, 2, 3, \dots) \quad (4)$$

and R_n represents the zeros of the Airy function along the negative abscissa. Appendix A describes the detailed derivation of equation (3) and its corresponding wave function. For $\ell \neq 0$, an exact solution does not exist; however, asymptotic solutions have been obtained to predict the energy levels. Among these, for arbitrary n and ℓ , is the Cornwall equation (ref. 17)

$$E_{n\ell} = \frac{3}{2} \left(\frac{\pi^2}{12} \right)^{1/3} \left[\frac{2k^2 (\hbar c)^2}{\mu c^2} \right]^{1/3} \left(\ell + 2n - \frac{1}{2} \right)^{2/3} \quad (5)$$

and the Sheehy and Von Baeyer equation (ref. 18)

$$E_{n_r\ell} = \left[\frac{2k^2(\hbar c)^2}{\mu c^2} \right]^{1/3} \left[3^{1/2} \frac{n_r + 1/2}{(\ell + 1/2)^{1/3}} + \frac{3}{2} \left(\ell + \frac{1}{2} \right)^{2/3} \right] \quad (6)$$

Note that in equation (5), $E_{n\ell}$ depends only on one combination of n and ℓ . Therefore, there is a degeneracy between the states (n, ℓ) and $(n - 1, \ell + 2)$. In equation (6), $E_{n_r\ell}$ with n_r defined in equation (4) is derived from a modification to the Bohr quantization of circular orbit with the added radial excitation being treated as a harmonic approximation and by the use of half-integer angular momentum quantum number which, when applied to the confined heavy mesons, yields a reasonably accurate spectrum.

Harmonic Potential

The three-dimensional harmonic potential is defined as

$$v(r) = \frac{1}{2}kr^2 \quad (7)$$

where k and r are defined in equation (2). The energy eigenvalues for arbitrary n and ℓ are given by

$$E_{n_r\ell} = \left(2n_r + \ell + \frac{3}{2} \right) \hbar c \left(\frac{k}{\mu c^2} \right)^{1/2} \quad (8)$$

Appendix B describes the detailed derivation of equation (8) and its corresponding wave function.

Spin Consideration

Since quarks have intrinsic spin of $1/2$, the following states are possible: $^1S_0, ^3S_0, ^1P_1, ^3P_0, ^3P_1, ^3P_2, ^1D_2, ^3D_1, ^3D_2, ^3D_3$, etc., where the notation is $(2S+1)L_J$, with S as the intrinsic spin of the meson, L as the orbital momentum, and J as the total meson spin. The notation S , P , and D correspond to the L values of 0, 1, and 2, respectively. Note that all these states are observed; however, in this work the concentration is on S and P states as shown in figures 1 and 2, with the corresponding masses of states obtained from reference 19. Note that spin effects are neglected, and average values for the 3P states shown in the last column (figs. 1 and 2) are included. This column is actually not the average but rather the average of the mass of the 3P_2 state plus error and 3P_0 state minus error, with the error quoted for the average 3P state being just made to extend up to the mass of the 3P_2 state plus error and down to the 3P_0 state minus error. Thus the mass and the error assigned to the average 3P state simply represent the range of the possible values of 3P_0 , 3P_1 , and 3P_2 states.

Results

The usual procedure in quarkonium research is to treat the charm or bottom mass $m_c c^2$ and $m_b c^2$ as free parameters, and based on the particular potential model in the nonrelativistic Schrödinger equation, fit E_n in equation (1) to two arbitrarily experimentally observed levels to determine k and V_0 . This means that there is a variety of possible fitting procedures that one could try. Here the charm quark mass, at a typically accepted value of $1500 \text{ MeV}/c^2$ (ref. 4), and the first two 3S_1 state levels of charmonium at 3096 and 3686 MeV/c^2 are used in equation (1) to obtain values of k and V_0 . With the first two levels fitted to the experimentally observed levels, k and V_0 are then used to predict other levels. With the linear potential model of equation (2), the calculated values for k and V_0 are $k = 1211 \text{ MeV}/\text{fm}$ and $V_0 = 690 \text{ MeV}$.

The results of this iterative procedure are shown in figures 3 and 4 and are compared with experimentally observed levels. Also included in figures 3 and 4 are the results of the harmonic potential model (eq. (7)), which were calculated by the authors in an earlier paper (ref. 16). In these figures, levels obtained by the linear potential model are labeled with an L , whereas levels obtained from the harmonic potential model are labeled with an H . Note that, for the linear potential model, good agreement is found in both the charmonium and bottomonium spectra for S and P levels. In almost all cases, the linear potential model produces a much more accurate spectrum than the harmonic potential model. The main advantage of the harmonic potential, however, is that the solution of the nonrelativistic Schrödinger equation is much more easily obtained than for either the linear or more exotic potential models. Therefore, the harmonic potential model has traditionally been used to obtain initial estimates for energy levels as a first try.

Recently, claims have been made that the sixth quark flavor, called top, has been found (refs. 20 and 21) with a mass in the range of 30 000 to 50 000 MeV/c^2 . Choosing the top mass to be 40 000 MeV/c^2 , calculations for toponium levels based on the linear potential model are presented in figure 5 along with predictions from the harmonic potential model.

As stated earlier, for $\ell \neq 0$, the nonrelativistic Schrödinger equation, with a linear potential, cannot be solved exactly. Thus asymptotic solutions of equations (5) and (6) were presented as good approximations for calculation of energy levels for arbitrary ℓ values. Table 1 presents the experimental results, exact results of equation (3), and approximate results

of equations (5) and (6) for charm, bottom, and top mesons, for the reader to draw conclusions on the applicability of equations (5) and (6). As is apparent from the table, in almost all cases, the Cornwall equation (eq. (5)) produces a more accurate spectrum than the Sheehy and Von Baeyer equation (eq. (6)). This is partially because equation (6) was derived based on the modification to the classic Bohr quantization of circular orbit, with the added radial excitation being treated in a harmonic approximation (perturbation), and equation (5) was derived based on purely asymptotic arguments without any added perturbative term and produces accurate energy levels even for toponium which is a highly debated topic in the present physics community.

Figures 6 through 11 present the linear and the harmonic potential wave function $U_{n_r\ell}[r]$ for charm, bottom, and top, in units of $\text{fm}^{-1/2}$ for $n_r = 0, 1, 2$, and 3 , and $\ell = 0$ and 1 , where the radial node n_r was defined by equation (4). For the linear case with $\ell = 0$, the wave function was derived from the exact solution of the nonrelativistic Schrödinger equation (eq. (A32)). For $\ell \neq 0$, a fourth-order Runge-Kutta scheme was used to numerically solve the nonrelativistic Schrödinger equation and obtain the energy levels (3P) and wave function. For the harmonic case with arbitrary ℓ , the wave function was derived from the exact solution of the nonrelativistic Schrödinger equation (eq. (B30)). In all figures, $U_{n_r\ell}[r]$ is plotted for the range $0.01 \leq r \leq 4 \text{ fm}$. As is apparent, the wave function is mass dependent and becomes larger in amplitude and narrower in width with increasing mass.

Finally, note that even though this and other works (refs. 10 and 22) yield agreement between theory and experiment with a linear potential model, there are other potential forms (such as the logarithmic form of Richardson (ref. 23)), which also give reasonable descriptions. This is because the main requirement is that the potential be confined. Good fits are also obtained by considering the form of the potential at intermediate distances (refs. 24 and 25), which represents a correction to the linear confinement form.

Concluding Remarks

Linear and harmonic potential models in conjunction with the nonrelativistic Schrödinger equation were used to obtain particle mass spectra of mesons as bound states of quarks. The main emphasis was on the linear potential where an exact solution of the S-state ($\ell = 0$) eigenvalues and eigenfunctions and an asymptotic solution for the higher order partial waves ($\ell > 0$) were obtained. Cornwall's asymptotic solution very accurately predicted all the energy levels including recently claimed toponium. Exact solutions of eigenvalue and eigenfunction for harmonic potential with arbitrary partial wave states were also obtained and were compared with linear potential results. For quark confinement, the linear potential is a much superior model, even though the solution of the nonrelativistic Schrödinger equation is more difficult to obtain.

NASA Langley Research Center
Hampton, VA 23665-5225
May 31, 1988

Appendix A

Derivation of Linear Potential Energy Levels and Wave Function

In this appendix, a detailed derivation of equation (3) and its corresponding wave function is presented.

The three-dimensional linear potential is given by

$$v(r) = kr \quad (\text{A1})$$

where k is the strength of potential and equation (A1) is subject to the limiting conditions

$$v(r) = \begin{cases} kr & (r > 0) \\ \infty & (r \leq 0) \end{cases} \quad (\text{A2})$$

Because of the spherical harmonic nature of this potential, the angular dependence of the wave function can be given by the spherical harmonics; therefore,

$$\psi(r, \theta, \phi) = R[r]Y_\ell^m(\theta, \phi) \quad (\text{A3})$$

where ψ is the complete wave function, and only the radial component $R[r]$ must be calculated, which satisfies the nonrelativistic Schrödinger equation according to

$$\left[\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} + \frac{\mu E}{\hbar^2} - \frac{\mu kr}{\hbar^2} \right] R[r] = 0 \quad (\text{A4})$$

An analytical solution of equation (A4) for $\ell \neq 0$ does not exist. However, for $\ell = 0$, equation (A4) becomes

$$\left[-\frac{d^2}{dr^2} - \frac{\mu E}{\hbar^2} + \frac{\mu kr}{\hbar^2} \right] R[r] = 0 \quad (\text{A5})$$

By defining new variables ξ and η

$$\xi = \frac{\mu E}{\hbar^2} \quad (\text{A6a})$$

$$\eta = \frac{\mu k}{\hbar^2} \quad (\text{A6b})$$

equation (A5) becomes

$$\left(-\frac{d^2}{dr^2} - \xi + \eta r \right) R[r] = 0 \quad (\text{A7})$$

The term $\eta r - \xi$ can be written as

$$\eta r - \xi = \beta \lambda \quad (\text{A8})$$

where β is a constant. Using equation (A8), after some algebra, equation (A7) becomes

$$\left(\frac{d^2}{d\lambda^2} - \frac{\beta^3}{\eta^2} \lambda \right) R[r] = 0 \quad (\text{A9})$$

Equation (A9) with the condition that $\beta^3 = \eta^2$ becomes

$$\left(\frac{d^2}{d\lambda^2} - \lambda\right) R[r] = 0 \quad (\text{A10})$$

The solution to equation (A10) is the Airy function which is closely related to the Bessel function of fraction order and, in integral form, is expressed as

$$Ai[\lambda] = \pi^{-1} \int_0^\infty \cos\left(\frac{q^3}{3} + \lambda q\right) dq \quad (\text{A11})$$

and

$$R[\lambda] = A Ai[\lambda] \quad (\text{A12})$$

where A , the normalization constant, is calculated later. With the boundary condition at $r = 0$,

$$R[\beta\lambda] = R[\eta r - \xi]|_{r=0} = 0 \quad (\text{A13})$$

we find

$$R[\beta\lambda] = R[-\xi] \quad (\text{A14})$$

or

$$R[\lambda] = R\left[-\frac{\xi}{\beta}\right] \quad (\text{A15})$$

and at boundary $r = 0$,

$$\lambda = -\frac{\xi}{\beta} = -\frac{\mu E}{\beta \hbar^2} \quad (\text{A16})$$

we find

$$\beta = -\frac{\mu E}{\lambda \hbar^2} \quad (\text{A17})$$

Using the condition $\beta^3 = \eta^2$ with $\eta = \frac{\mu k}{\hbar^2}$ gives

$$\left(-\frac{\mu E}{\lambda \hbar^2}\right)^3 = \left(\frac{\mu k}{\hbar^2}\right)^2 \quad (\text{A18})$$

or

$$\lambda = -\left(\frac{\mu}{\hbar^2 k^2}\right)^{1/3} E \quad (\text{A19})$$

Therefore, eigenvalues of E are proportional to the roots of Airy function, and thus at $r = 0$,

$$R\left[-\left(\frac{\mu}{\hbar^2 k^2}\right)^{1/3} E\right] = Ai\left[-\left(\frac{\mu}{\hbar^2 k^2}\right)^{1/3} E\right] = 0 \quad (\text{A20})$$

or

$$E_{n_r} = \left(\frac{\hbar^2 k^2}{\mu}\right)^{1/3} |R_n| \quad (n_r = n - 1 = 0, 1, 2, \dots) \quad (\text{A21})$$

Equation (A21) gives the energy levels in terms of n_r and R_n (zeros of Airy function). We can rewrite this result in terms of our units as

$$E_{n_r} = \left[\frac{k^2 (\hbar c)^2}{\mu c^2} \right]^{1/3} |R_n| \quad (\text{A22})$$

Now, since

$$\beta \lambda = \eta r - \xi = -\frac{\mu}{\hbar^2} \left[\left(\frac{\hbar^2 k^2}{\mu} \right)^{1/3} R_n \right] + \frac{\mu k}{\hbar^2} r \quad (\text{A23})$$

or

$$\beta \left[\frac{\hbar^4}{\mu^2 k^2} \right]^{1/3} \lambda = -R_n + \left(\frac{\mu k}{\hbar^2} \right)^{1/3} r \quad (\text{A24})$$

which means $R_{n_r}[r]$ can be expressed as

$$R_{n_r}[r] = A \operatorname{Ai} \left[-R_n + \left(\frac{\mu k}{\hbar^2} \right)^{1/3} r \right] \quad (\text{A25})$$

where the normalization constant A is

$$A = \left[\int_{-R_n}^{\infty} R_{n_r}^2[r] dr \right]^{-1/2} = \left[\frac{\mu}{\pi^{3/2} k^{1/2} \hbar^2} \right]^{1/3} \quad (\text{A26})$$

Therefore the wave function $R_{n_r}[r]$ is

$$R_{n_r}[r] = \left[\frac{\mu}{\pi^{3/2} k^{1/2} \hbar^2} \right]^{1/3} \operatorname{Ai} \left[-R_n + \left(\frac{\mu k}{\hbar^2} \right)^{1/3} r \right] \quad (\text{A27})$$

In order to evaluate equation (A27) numerically, use reference 26 and the identity

$$\operatorname{Ai}[z] = c_1 f[z] - c_2 g[z] \quad (\text{A28})$$

where

$$f[z] = \sum_0^{\infty} 3^k \left(\frac{1}{3} \right)_k \frac{z^{3k}}{(3k)!}$$

$$g[z] = \sum_0^{\infty} 3^k \left(\frac{2}{3} \right)_k \frac{z^{3k+1}}{(3k+1)!}$$

$$c_1 = 3^{-2/3} \Gamma[2/3]$$

$$c_2 = 3^{-1/3} \Gamma[1/3]$$

Equation (A27) can be written as

$$R_{n_r}[r] = \left\{ c_1 f \left[-R_n + \left(\frac{\mu k}{\hbar^2} \right)^{1/3} r \right] - c_2 g \left[-R_n + \left(\frac{\mu k}{\hbar^2} \right)^{1/3} r \right] \right\} \left(\frac{\mu}{\pi^{3/2} k^{1/2} \hbar^2} \right)^{1/3} \quad (\text{A29})$$

Defining the reduced wave function $U[r]$ as

$$U[r] = \frac{R[r]}{r} \quad (\text{A30})$$

Equation (A29) becomes

$$U_{n_r}[r] = \frac{\left\{ c_1 f \left[-R_n + \left(\mu k / \hbar^2 \right)^{1/3} r \right] - c_2 g \left[-R_n + \left(\mu k / \hbar^2 \right)^{1/3} r \right] \right\} \left(\frac{\mu}{\pi^{3/2} k^{1/2} \hbar^2} \right)^{1/3}}{r} \quad (\text{A31})$$

With \hbar in units of MeV-s and c in units of fm/s, equation (A31) becomes

$$U_{n_r}[r] = \frac{\left\{ c_1 f \left[-R_n + \left(\mu c^2 k / (\hbar c)^2 \right)^{1/3} r \right] - c_2 g \left[-R_n + \left(\mu c^2 k / (\hbar c)^2 \right)^{1/3} r \right] \right\} \left[\frac{\mu c^2}{\pi^{3/2} k^{1/2} (\hbar c)^2} \right]^{1/3}}{r} \quad (\text{A32})$$

Equation (A32) in units of $\text{fm}^{1/2}$ along with equation (A22) describes the wave function and energy levels of linear potential with $\ell = 0$ completely.

Appendix B

Derivation of Harmonic Potential Energy Levels and Wave Function

In this appendix, a detailed derivation of equation (8) and its corresponding wave function is presented.

The three-dimensional harmonic potential is given by

$$v(r) = \frac{1}{2}kr^2 \quad (\text{B1})$$

where

$$k = \mu\omega^2 \quad (\text{B2})$$

with

$$\mu = \frac{m_q m_{\bar{q}}}{m_q + m_{\bar{q}}} = \frac{m_q}{2}$$

being the $q\bar{q}$ pair reduced mass and

$$\omega = \left(\frac{k}{\mu}\right)^{1/2}$$

being the angular frequency of the oscillation with k as the spring constant. In appendix A, it was shown that the complete wave function ψ can be written as the product of a radial component $R[r]$ and a spherical harmonics $Y_\ell^m(\theta, \phi)$ as

$$\psi(r, \theta, \phi) = R[r]Y_\ell^m(\theta, \phi) \quad (\text{B3})$$

and only the radial component $R[r]$ must be calculated which satisfies the nonrelativistic Schrödinger equation according to

$$\left[\frac{d^2}{dr^2} - \frac{\ell(\ell+1)}{r^2} + \frac{\mu E}{\hbar^2} - \frac{\mu^2 \omega^2 r^2}{\hbar^2} \right] R[r] = 0 \quad (\text{B4})$$

By defining new variables ξ and η

$$\xi = \left(\frac{\hbar}{\mu\omega}\right)^{1/2} \quad (\text{B5a})$$

$$\eta = \xi^{-1}r \quad (\text{B5b})$$

equation (B4) becomes

$$\left[\frac{d^2}{d\eta^2} - \frac{\ell(\ell+1)}{\eta^2} + \frac{2E}{\hbar\omega} - \eta^2 \right] R[\eta] = 0 \quad (\text{B6})$$

A solution for $R[\eta]$ in equation (B6) must be found subject to the following limiting conditions:

$$R[\eta] = \left\{ \begin{array}{ll} 0 & (\eta \rightarrow 0) \\ 0 & (\eta \rightarrow \infty) \end{array} \right\} \quad (\text{B7})$$

A reasonable choice for $R[\eta]$ can then be

$$R[\eta] = \left\{ \begin{array}{ll} \eta^{\ell+1} & (\eta \rightarrow 0) \\ \exp\left(-\frac{1}{2}\eta^2\right) & (\eta \rightarrow \infty) \end{array} \right\} \quad (\text{B8})$$

therefore,

$$R[\eta] = \eta^{\ell+1} \exp\left(-\frac{1}{2}\eta^2\right) P[\eta] \quad (\text{B9})$$

Replacing $R[\eta]$ in equation (B6) with the one in equation (B9), equation (B6) becomes

$$\frac{d^2 P[\eta]}{d\eta^2} + 2\left(\frac{\ell+1}{\eta} - \eta\right) \frac{dP[\eta]}{d\eta} + \left[\frac{2E}{\hbar\omega} - (2\ell+3)\right] P[\eta] = 0 \quad (\text{B10})$$

By defining a new variable f as $f = \eta^2$ and after some algebra, equation (B10) becomes

$$\frac{d^2 P[f]}{df^2} + \frac{1}{2f} [(2\ell+3) - 2f] \frac{dP[f]}{df} + \frac{1}{4f} \left[\frac{2E}{\hbar\omega} - (2\ell+3)\right] P[f] = 0 \quad (\text{B11})$$

or

$$\frac{d^2 P[f]}{df^2} + \frac{1}{f} \left[\left(\ell + \frac{3}{2}\right) - f\right] \frac{dP[f]}{df} + \frac{1}{2f} \left[\left(\ell + \frac{3}{2}\right) - \frac{E}{\hbar\omega}\right] P[f] = 0 \quad (\text{B12})$$

If

$$x = \ell + \frac{3}{2} \quad (\text{B13a})$$

$$y = \frac{1}{2} \left[\left(\ell + \frac{3}{2}\right) - \frac{E}{\hbar\omega}\right] \quad (\text{B13b})$$

Then equation (B12) becomes

$$\frac{d^2 P[f]}{df^2} + \frac{1}{f}(x-f) \frac{dP[f]}{df} - \frac{y}{f} P[f] = 0 \quad (\text{B14})$$

which is the standard confluent hypergeometric equation, and

$$P[\eta] = P\left[\frac{1}{2}\left(\ell + \frac{3}{2} - \frac{E}{\hbar\omega}\right), \ell + \frac{3}{2}; \eta^2\right] \quad (\text{B15})$$

Equation (B15) indicates that in equation (B9) when $\eta \rightarrow \infty$ and $R[\eta] \rightarrow \exp\left(-\frac{1}{2}\eta^2\right)$, equation (B15) can be expanded in a finite number of terms. This is true when y in equation (B13b) is zero or a negative integer; therefore,

$$\frac{1}{2} \left[\left(\ell + \frac{3}{2}\right) - \frac{E}{\hbar\omega}\right] = -n_r \quad (n_r = 0, 1, 2, 3, \dots) \quad (\text{B16})$$

or

$$E_{n_r \ell} = \left(2n_r + \ell + \frac{3}{2}\right) \hbar\omega \quad (\text{B17a})$$

and with equation (B2), $E_{n_r \ell}$ becomes

$$E_{n_r \ell} = \left(2n_r + \ell + \frac{3}{2}\right) \hbar c \left(\frac{k}{\mu c^2}\right)^{1/2} \quad (\text{B17b})$$

which is simply the energy levels in terms of n_r and ℓ and expressed in units of MeV. Now, equation (B9) can be written as

$$R_{n_r \ell}[\eta] = A \eta^{\ell+1} \exp\left(-\frac{1}{2}\eta^2\right) P\left[-n_r, \ell + \frac{3}{2}; \eta^2\right] \quad (\text{B18})$$

where A , the normalization constant, is calculated later.

In equation (B15), $P\left[-n_r, \ell + \frac{3}{2}; \eta^2\right]$ is related to the associated Laguerre polynomials by

$$L_{n_r}^{\ell+1/2}[f] = \frac{\left(\Gamma\left[n_r + \ell + \frac{3}{2}\right]\right)^2}{n_r! \Gamma\left[\ell + \frac{3}{2}\right]} P\left[-n_r, \ell + \frac{3}{2}; f\right] \quad (\text{B19})$$

Therefore, equation (B18) becomes

$$R_{n_r \ell} = A \eta^{\ell+1} \exp\left(-\frac{1}{2}\eta^2\right) L_{n_r}^{\ell+1/2}[\eta^2] \quad (\text{B20})$$

where $L_{n_r}^{\ell+1/2}[\eta^2]$ is defined as

$$L_{n_r}^{\ell+1/2}[\eta^2] = \sum_{m=0}^{n_r} \frac{(n_r + \ell + \frac{1}{2})!}{(n_r - m)! (m + \ell + \frac{1}{2})!} \frac{(-\eta^2)^m}{m!} \quad (\text{B21})$$

From a computational point of view, the sum in equation (B21) is cumbersome to calculate. It however can be recast by defining two new variables N and M as

$$N = n_r + \ell \quad (\text{B22a})$$

$$M = m + \ell \quad (\text{B22b})$$

Therefore, $n_r = N - \ell$, $m = M - \ell$, $n_r - m = (N - \ell) - (M - \ell) = N - M$, and equation (B21) in terms of ℓ , M , and N can be written as

$$L_{n_r}^{\ell+1/2}[\eta^2] = \sum_{M=\ell}^{M=N} \frac{(N + \frac{1}{2})!}{(N - M)! (M + \frac{1}{2})!} \frac{(-\eta^2)^{M-\ell}}{(M - \ell)!} \quad (\text{B23})$$

Now, by using the identity

$$\left(k + \frac{1}{2}\right)! = \frac{(2k + 1)!}{2^{2k+1} k!} (\pi)^{1/2} \quad (\text{B24})$$

Equation (B23) becomes

$$L_{n_r}^{\ell+1/2}[\eta^2] = \sum_{M=\ell}^{M=N} \frac{\left[(2N + 1)! / (2^{2N+1} N!)\right]}{[(N - M)!] [(2M + 1)! / (2^{2M+1} M!)]} \frac{(-\eta^2)^{M-\ell}}{(M - \ell)!} \quad (\text{B25})$$

which simplifies to

$$L_{n_r}^{\ell+1/2}[\eta^2] = \sum_{M=\ell}^{M=N} \frac{[(2N + 1)!] (2^{2M+1} M!)}{[(N - M)!] [(2M + 1)!] (2^{2N+1} N!)} \frac{(-\eta^2)^{M-\ell}}{(M - \ell)!} \quad (\text{B26})$$

Therefore, by using the results of equations (B22) and (B26), equation (B20) becomes

$$R_{n_r \ell} = A \eta^{\ell+1} \exp\left(-\frac{1}{2}\eta^2\right) \sum_{M=\ell}^{M=N} \frac{(2N + 1)! (2^{2M+1} M!)}{[(N - M)!] [(2M + 1)!] (2^{2N+1} N!)} \frac{(-\eta^2)^{M-\ell}}{(M - \ell)!} \quad (\text{B27})$$

where A can be calculated according to

$$\int_0^\infty R_{n\ell}^2[r] dr = 1 \quad (\text{B28})$$

and by using equations (B5), (B22), (B27), and (B28), the radial wave function $R_{n\ell}[r]$ becomes

$$\begin{aligned} R_{n\ell} = & \left(\frac{\hbar c}{\sqrt{\mu c^2 k}} \right)^{-1/4} \left(\frac{2n!}{\Gamma \left[n + \ell + \frac{3}{2} \right]} \right)^{1/2} \left[\left(\frac{\hbar c}{\sqrt{\mu c^2 k}} \right)^{-1/2} r \right]^{\ell+1} \exp \left[-\frac{1}{2} \left(\frac{\hbar c}{\sqrt{\mu c^2 k}} \right)^{-1} r^2 \right] \\ & \times \sum_{M=\ell}^{M=N} \frac{[(2N+1)!](2^{2M+1}M!)}{[(N-M)!][(2M+1)!](2^{2N+1}N!)} \frac{\left[-\left(\hbar c / \sqrt{\mu c^2 k} \right)^{-1} r^2 \right]^{M-\ell}}{(M-\ell)!} \end{aligned} \quad (\text{B29})$$

In equation (B29), $R_{n\ell}$ is in units of $\text{fm}^{-3/2}$, and, finally by using equation (A30) with \hbar in units of MeV-s and c in units of fm/s, equation (B29) becomes

$$\begin{aligned} U_{n\ell}[r] = & \left(\frac{\hbar c}{\sqrt{\mu c^2 k}} \right)^{-3/4} \left[\frac{2n!}{\Gamma \left(n + \ell + \frac{3}{2} \right)} \right]^{1/2} \left[\left(\frac{\hbar c}{\sqrt{\mu c^2 k}} \right)^{-1/2} r \right]^\ell \exp \left[-\frac{1}{2} \left(\frac{\hbar c}{\sqrt{\mu c^2 k}} \right)^{-1} r^2 \right] \\ & \times \sum_{M=\ell}^{M=N} \frac{[(2N+1)!](2^{2M+1}M!)}{[(N-M)!][(2M+1)!](2^{2N+1}N!)} \frac{\left[-\left(\hbar c / \sqrt{\mu c^2 k} \right)^{-1} r^2 \right]^{M-\ell}}{(M-\ell)!} \end{aligned} \quad (\text{B30})$$

Equation (B30) in units of $\text{fm}^{-1/2}$ along with equation (B17b) describes the wave function and energy levels of harmonic potential completely.

References

1. Antipppa, Adel F.; and Phares, Alain J.: The Linear Potential: A Solution in Terms of Combinatorics Functions. *J. Math. Phys.*, vol. 19, no. 1, Jan. 1978, pp. 308-319.
2. Critchfield, C. L.: Scalar Binding of Quarks. *Phys. Review*, vol. D12, no. 3, Aug. 1, 1975, pp. 923-925.
3. DeGrand, T.; Jaffe, R. L.; Johnson, K.; and Kiskis, J.: Masses and Other Parameters of the Light Hadrons. *Phys. Review*, vol. D12, no. 7, Oct. 1, 1975, pp. 2060-2076.
4. Gasiorowicz, Stephen; and Rosner, Jonathan L.: Hadron Spectra and Quarks. *American J. Phys.*, vol. 49, no. 10, Oct. 1981, pp. 954-984.
5. Hey, Anthony J. G.; and Kelly, Robert L.: Baryon Spectroscopy. *Phys. Rep.*, vol. 96, nos. 2 & 3, June 1983, pp. 71-204.
6. Hey, A. J. G.; and Morgan, D.: Meson Spectroscopy. *Rep. Prog. Phys.*, vol. 41, no. 5, May 1978, pp. 675-760.
7. Appelquist, Thomas; Barnett, R. Michael; and Lane, Kenneth: Charm and Beyond. *Annual Review of Nuclear and Particle Science, Volume 28*, J. D. Jackson, Harry E. Gove, and Roy F. Schwitters, eds., Annual Reviews Inc., 1978, pp. 387-499.
8. Close, F. E.: *An Introduction to Quarks and Partons*. Academic Press, Inc., 1979.
9. Halzen, Francis; and Martin, Alan D.: *Quarks and Leptons: An Introductory Course in Modern Particle Physics*. John Wiley & Sons, Inc., c.1984.
10. Eichten, E.; Gottfried, K.; Kinoshita, T.; Lane, K. D.; and Yan, T. M.: Charmonium: Comparison With Experiment. *Phys. Review*, vol. D21, no. 1, Jan. 1, 1980, pp. 203-233.
11. Isgur, Nathan; and Karl, Gabriel: P-Wave Baryons in the Quark Model. *Phys. Review*, vol. D18, no. 11, Dec. 1, 1978, pp. 4187-4205.
12. Faiman, David; and Hendry, Archibald W.: Harmonic-Oscillator Model for Baryons. *Phys. Review*, vol. 173, no. 5, Sept. 25, 1968, pp. 1720-1729.
13. Feynman, R.; Kislinger, M.; and Ravndal, F.: Current Matrix Elements From a Relativistic Quark Model. *Phys. Review*, vol. D3, no. 11, June 1, 1971, pp. 2706-2732.
14. Lipes, Richard G.: Electromagnetic Excitations of the Nucleon in a Relativistic Quark Model. *Phys. Review*, vol. D5, no. 11, June 1, 1972, pp. 2849-2863.
15. Osborne, L. S.: Relativistic Oscillator Model of High-Energy Interactions. *Phys. Review Lett.*, vol. 34, no. 2, Jan. 13, 1975, pp. 106-110.
16. Norbury, J. W.; Badavi, F. F.; and Townsend, L. W.: Predicting Charmonium and Bottomonium Spectra With a Quark Harmonic Oscillator. *American J. Phys.*, vol. 54, no. 11, Nov. 1986, pp. 1031-1034.
17. Cornwall, J. M.: What is the Relativistic Generalization of a Linearly Rising Potential? *Nucl. Phys.*, vol. B128, no. 1, Sept. 19, 1977, pp. 75-92.
18. Sheehy, Brian, and Von Baeyer, Hans C.: Modified Bohr Quantization of Charmonium. *American J. Phys.*, vol. 49, no. 5, May 1981, pp. 429-431.
19. Particle Data Group (C. G. Wohl, R. N. Cahn, et al.): Review of Particle Properties. *Reviews Modern Phys.*, vol. 56, no. 2, pt. II, Apr. 1984, pp. S1-S304.
20. UA2 Collaboration (P. Bagnaia, M. Banner, et al.): Observation of Electrons Produced in Association With Hard Jets and Large Missing Transverse Momentum in $p\bar{p}$ Collisions at $\sqrt{s} = 540$ GeV. *Phys. Lett.*, vol. 139B, no. 1, 2, May 3, 1984, pp. 105-114.
21. UA1 Collaboration (G. Arnison, O. C. Allkofer, et al.): Experimental Observation of Events With Large Missing Transverse Energy Accompanied by a Jet or a Photon (S) in $p\bar{p}$ Collisions at $\sqrt{s} = 540$ GeV. *Phys. Lett.*, vol. 139B, no. 1, 2, May 3, 1984, pp. 115-125.
22. Martin, A.: A Simultaneous Fit of $b\bar{b}$, $c\bar{c}$, $s\bar{s}$ (bcs Pairs) and $c\bar{s}$ Spectra. *Phys. Lett.*, vol. 100B, no. 6, Apr. 23, 1981, pp. 511-514.
23. Richardson, John L.: The Heavy Quark Potential and the T, J/ ψ Systems. *Phys. Lett.*, vol. 82B, no. 2, Mar. 26, 1979, pp. 272-274.
24. Celmaster, William; Georgi, Howard; and Machacek, Marie: Potential Model of Meson Masses. *Phys. Review*, vol. D17, no. 3, Feb. 1, 1978, pp. 879-885.
25. Buchmüller, W.; and Tye, S.-H. H.: Quarkonia and Quantum Chromodynamics. *Phys. Review*, vol. D24, no. 1, July 1, 1981, pp. 132-156.
26. Abramowitz, Milton; and Stegun, Irene A., eds.: *Handbook of Mathematical Functions With Formulas, Graphs, and Mathematical Tables*. NBS Appl. Math. Ser. 55, U.S. Dep. of Commerce, June 1964.

Table 1. Experimental and Theoretical Resonances for Linear Potential

n	n_r	ℓ	Experimental resonances, MeV/c^2	Calculated resonances, MeV/c^2		
				Linear potential (eq. (3))	Sheehy and Von Baeyer (eq. (6))	Cornwall (eq. (5))
Charmonium						
1	0	0	3 096	3 095	3 172	3 091
2	1	0	3 686	3 684	4 098	3 684
3	2	0	4 030	4 166	5 023	4 167
4	3	0	4 415	4 592	5 948	4 594
1	0	1	3 485	3 441	3 463	3 408
2	1	1		3 953	4 105	3 935
3	2	1		4 399	4 747	4 386
4	3	1		4 802	5 388	4 792
Bottomonium						
1	0	0	9 460	9 633	9 687	9 632
2	1	0	10 023	10 030	10 311	10 032
3	2	0	10 355	10 355	10 934	10 358
4	3	0	10 573	10 643	11 558	10 645
1	0	1	9 892	9 868	9 883	9 846
2	1	1	10 250	10 213	10 316	10 201
3	2	1		10 514	10 748	10 505
4	3	1		10 785	11 181	10 779
Toponium						
1	0	0		79 533	79 598	79 571
2	1	0		79 738	79 908	79 770
3	2	0		79 902	80 218	79 931
4	3	0		80 047	80 527	80 074
1	0	1		79 685	79 696	79 677
2	1	1		79 855	79 910	79 854
3	2	1		80 002	80 125	80 005
4	3	1		80 137	80 340	80 141

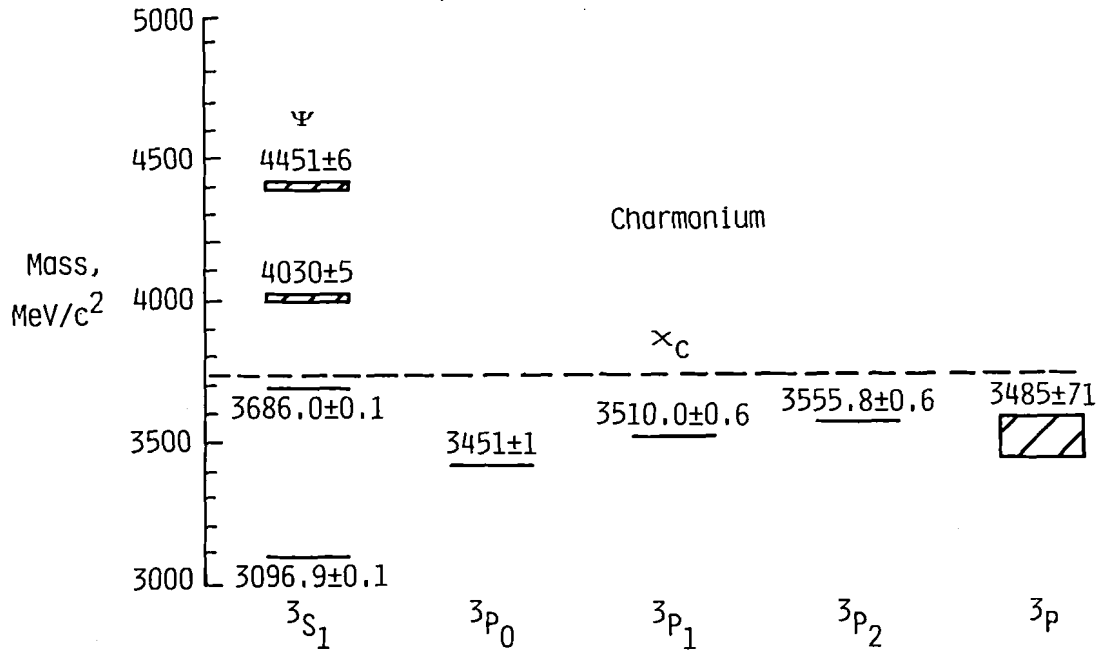


Figure 1. Experimental energy level spectra of charmonium. Numbers on each level are exact values in MeV/c^2 . Values are taken from reference 19.

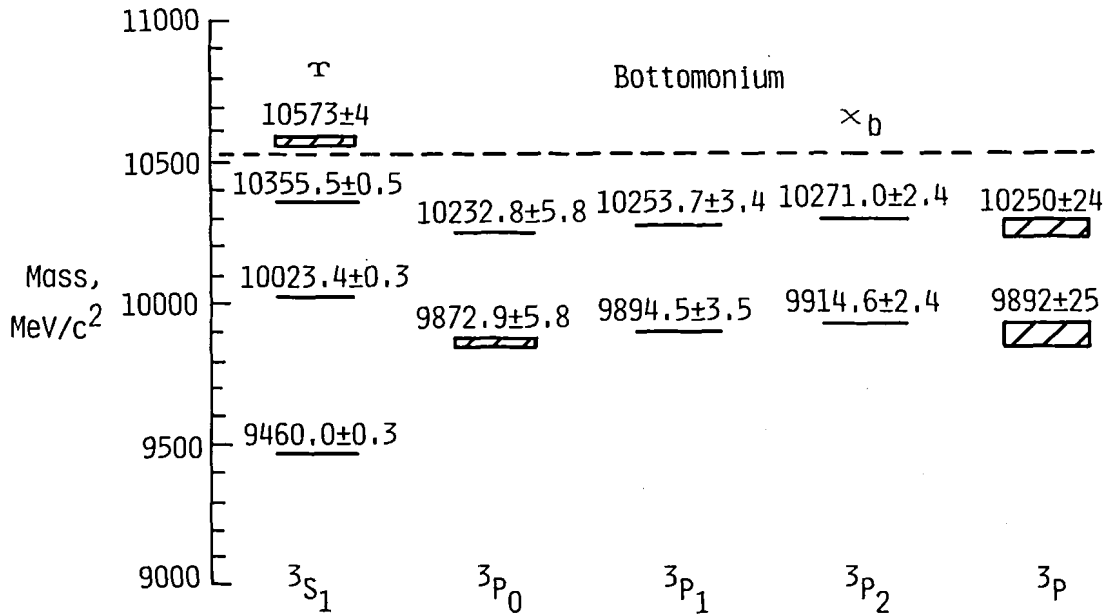


Figure 2. Experimental energy level spectra of bottomonium. Numbers on each level are exact values in MeV/c^2 . Values are taken from reference 19.

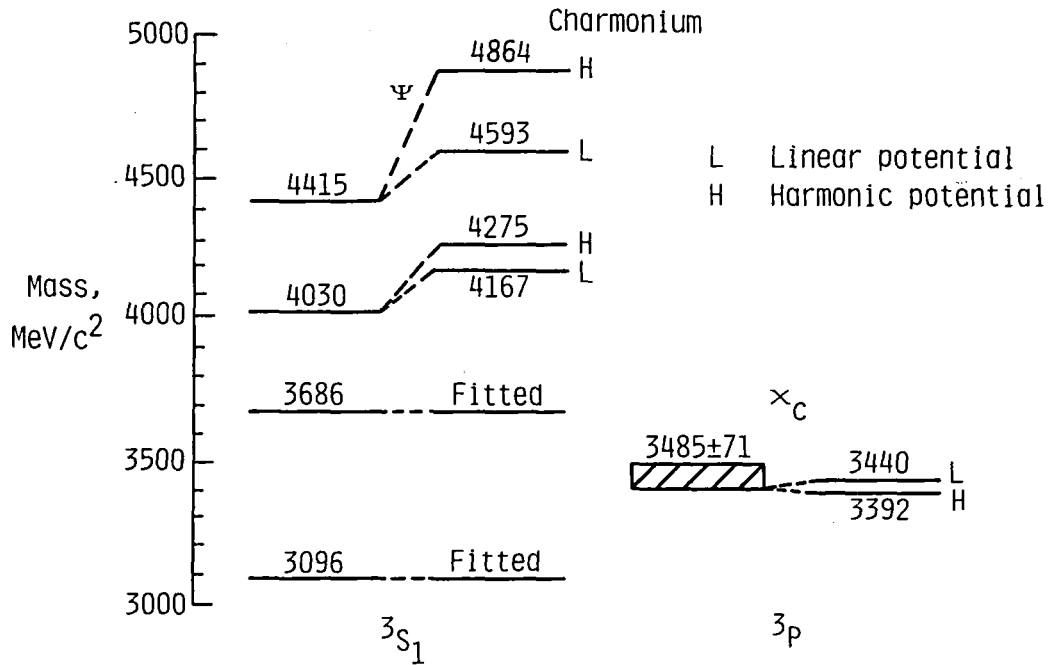


Figure 3. Calculations of linear and harmonic potential model levels versus experimental levels for charmonium spectrum.

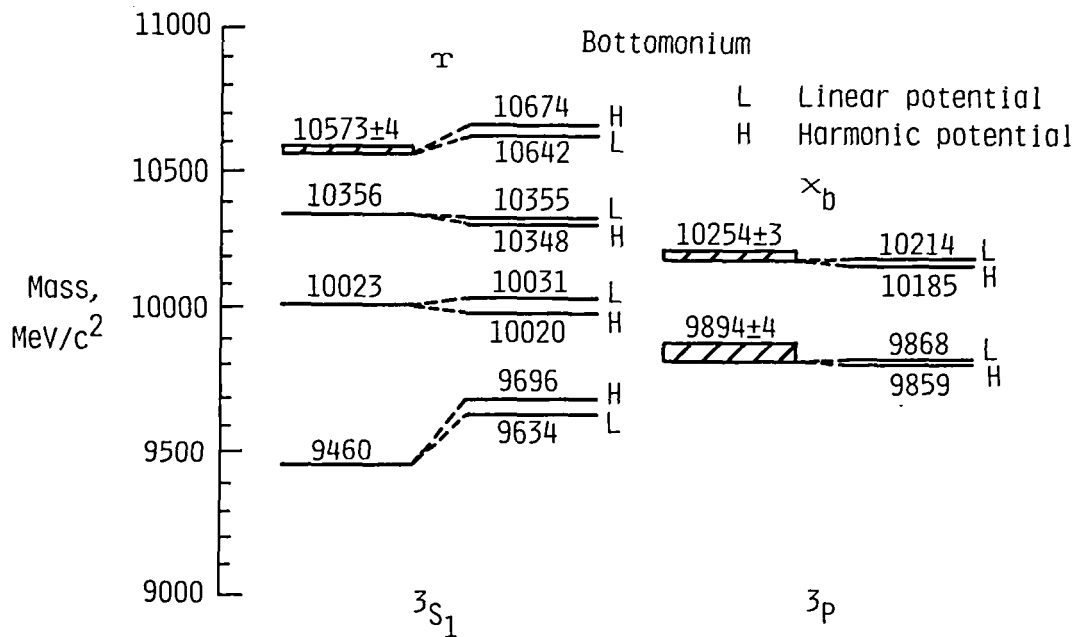


Figure 4. Calculations of linear and harmonic potential model levels versus experimental levels for bottomonium spectrum.

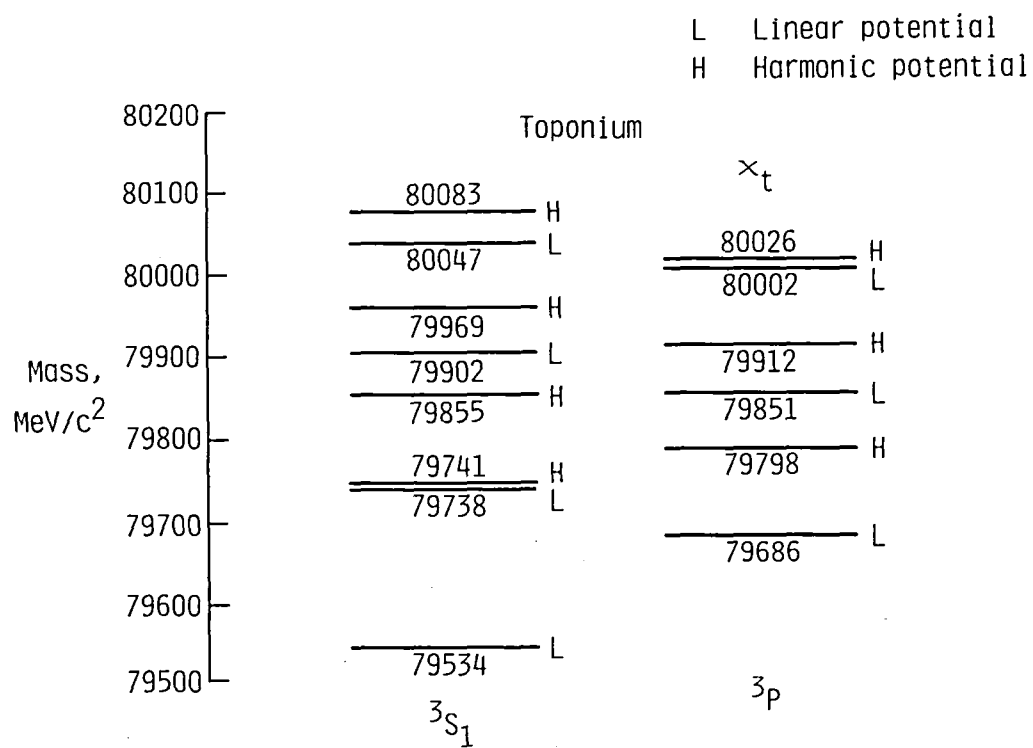


Figure 5. Calculated toponium spectrum.

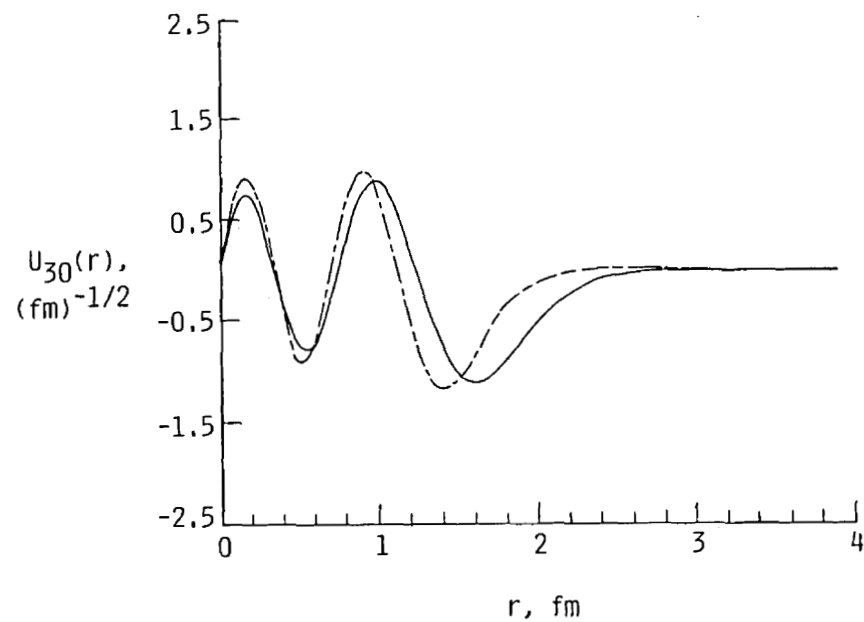
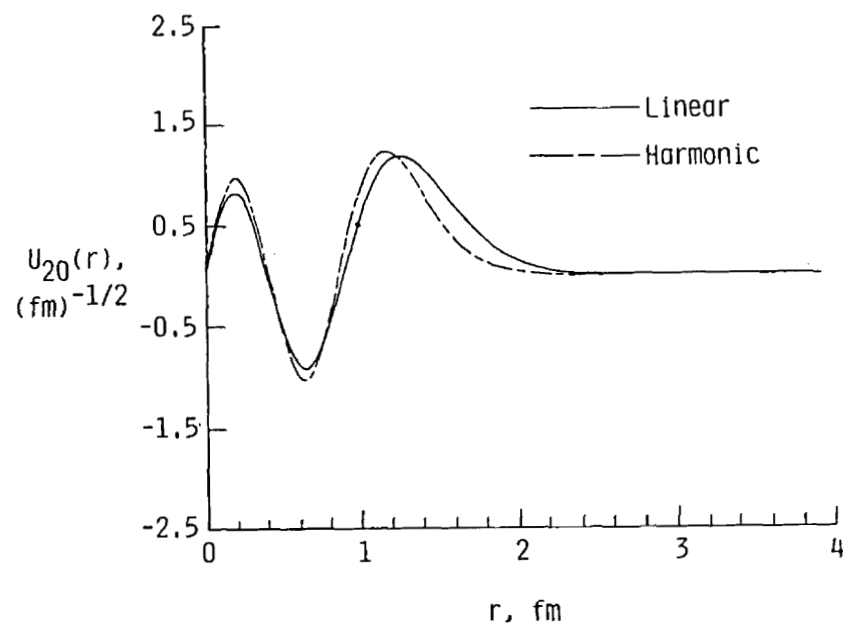
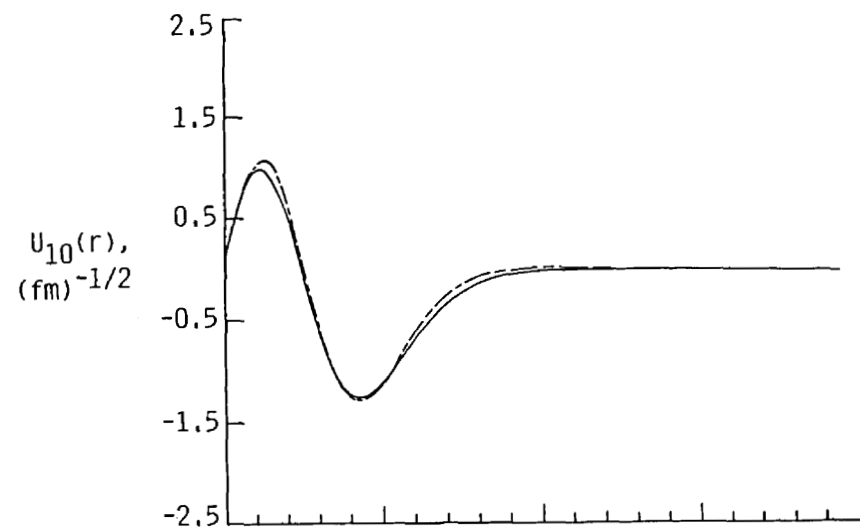
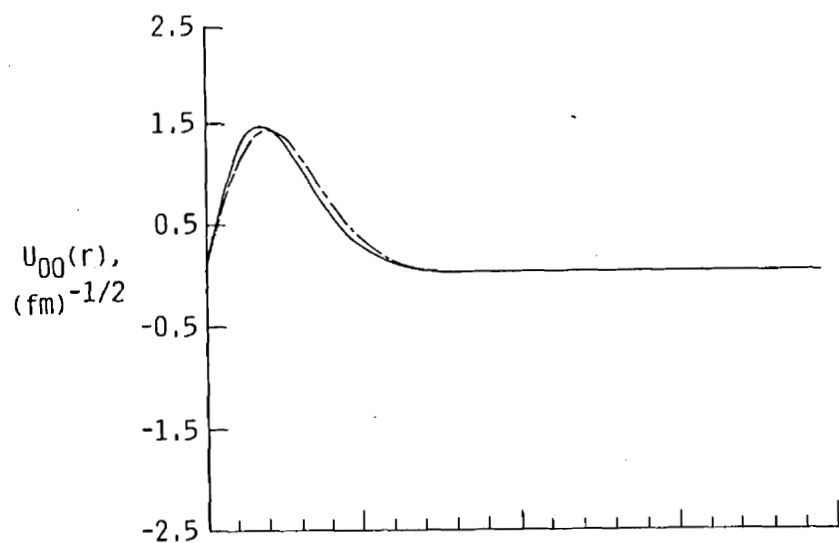


Figure 6. Charm radial wave function $U_{n_r\ell}(r)$ for $n_r = 0, 1, 2$, and 3 and $\ell = 0$.

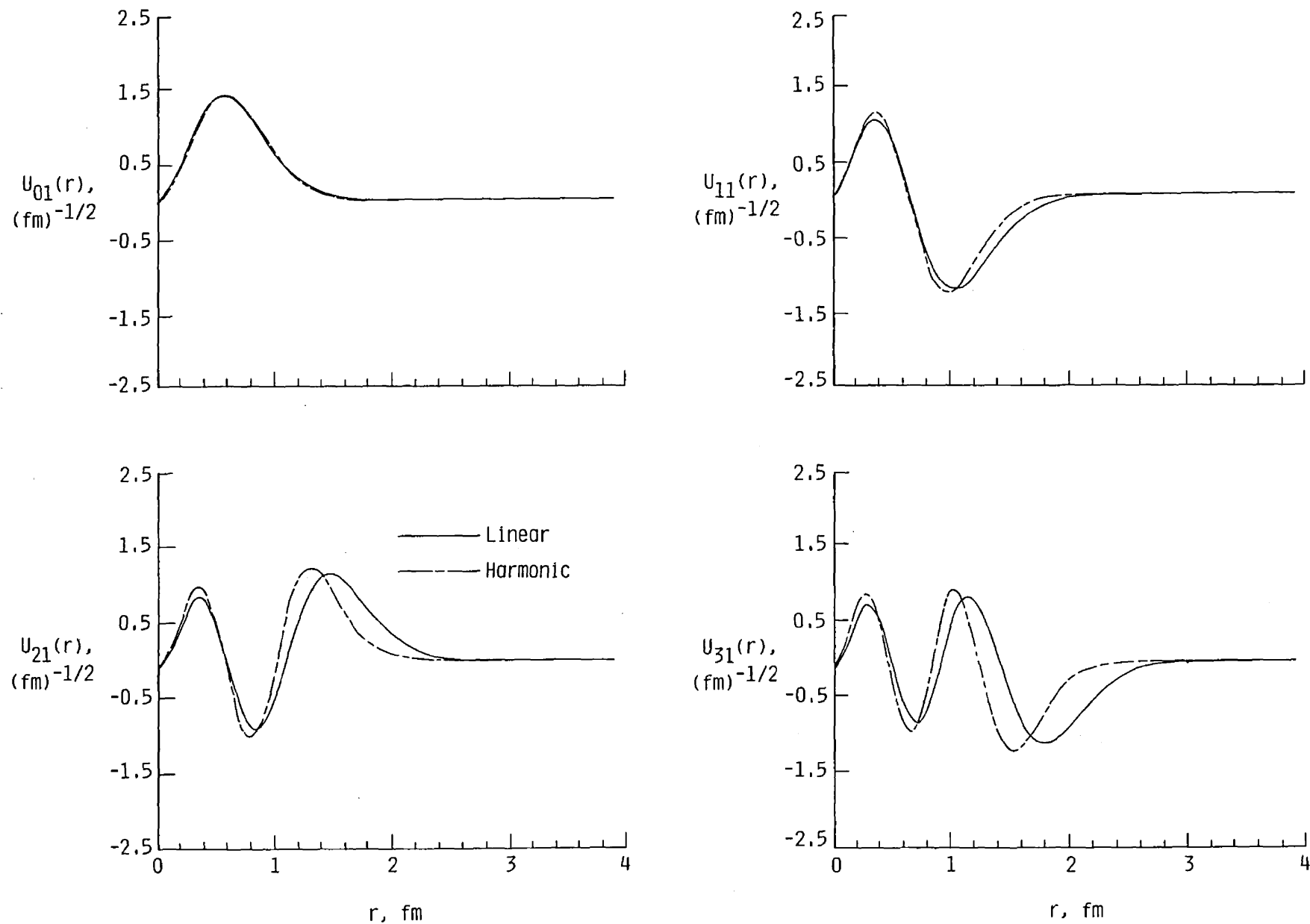


Figure 7. Charm radial wave function $U_{n_r \ell}(r)$ for $n_r = 0, 1, 2$, and 3 and $\ell = 1$.

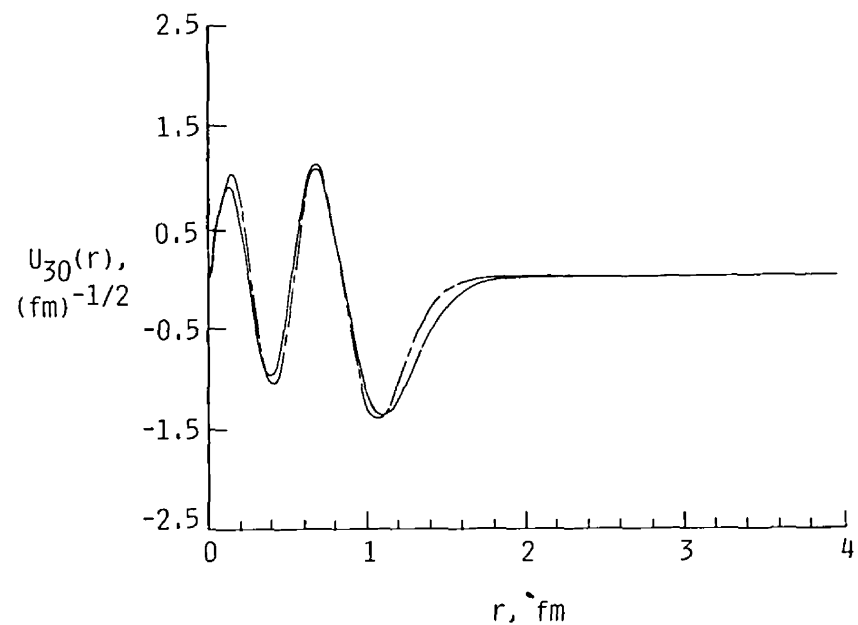
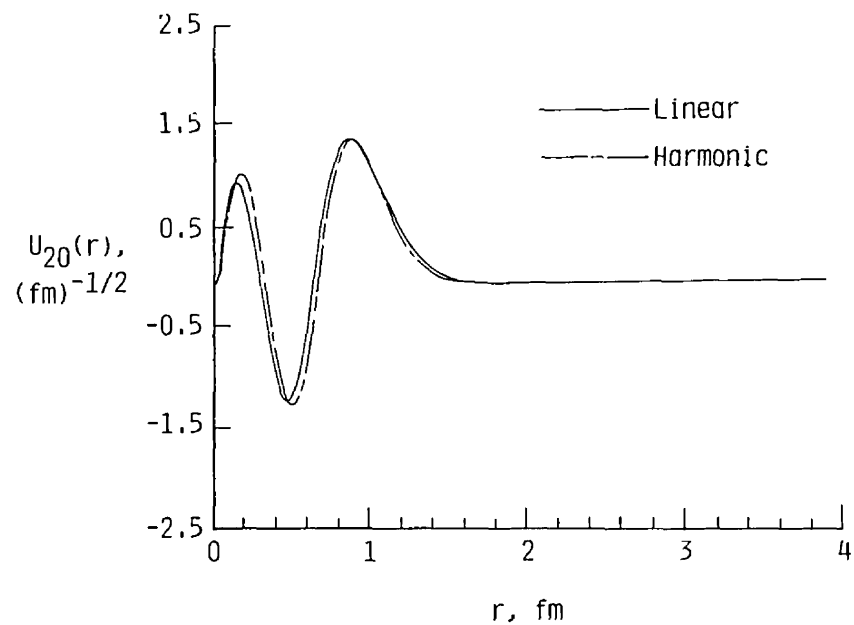
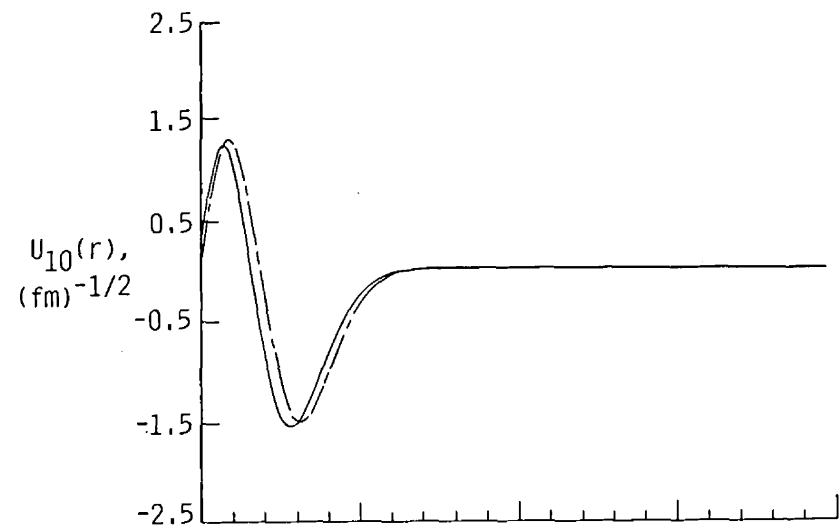
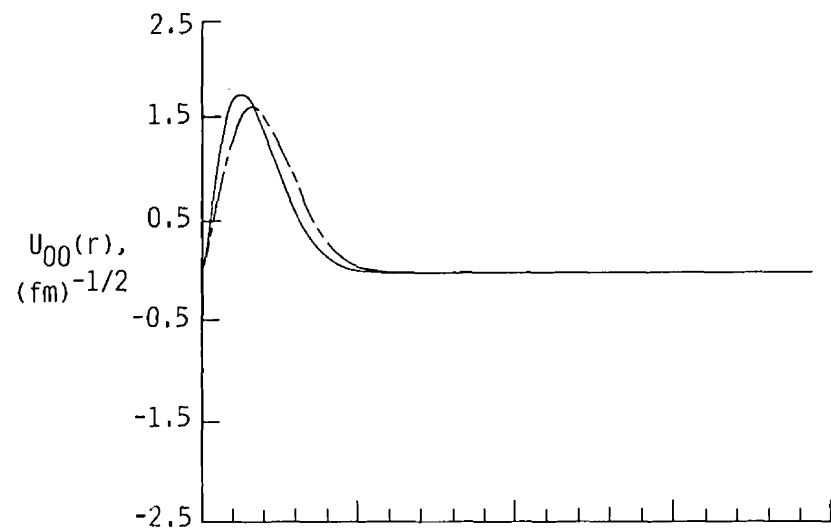


Figure 8. Bottom radial wave function $U_{n_r\ell}(r)$ for $n_r = 0, 1, 2$, and 3 and $\ell = 0$.

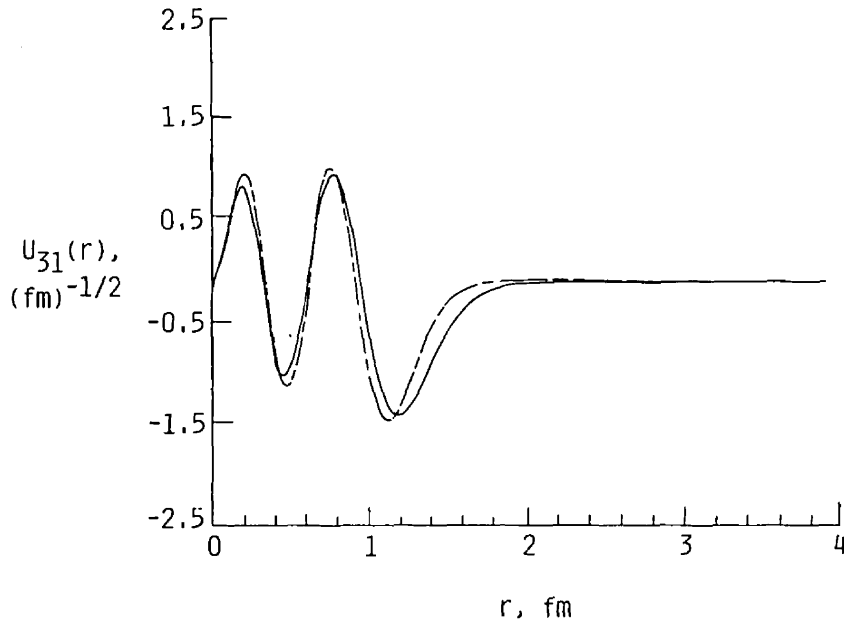
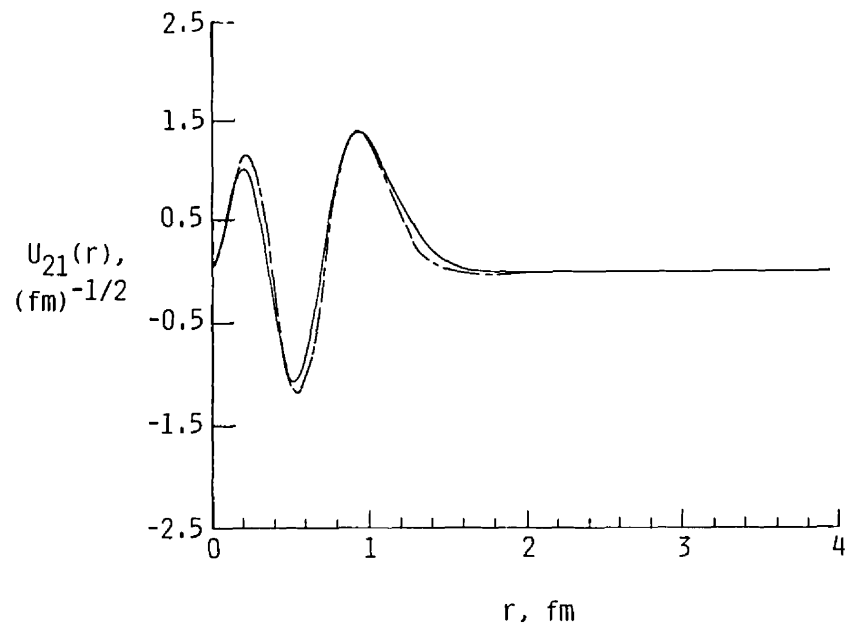
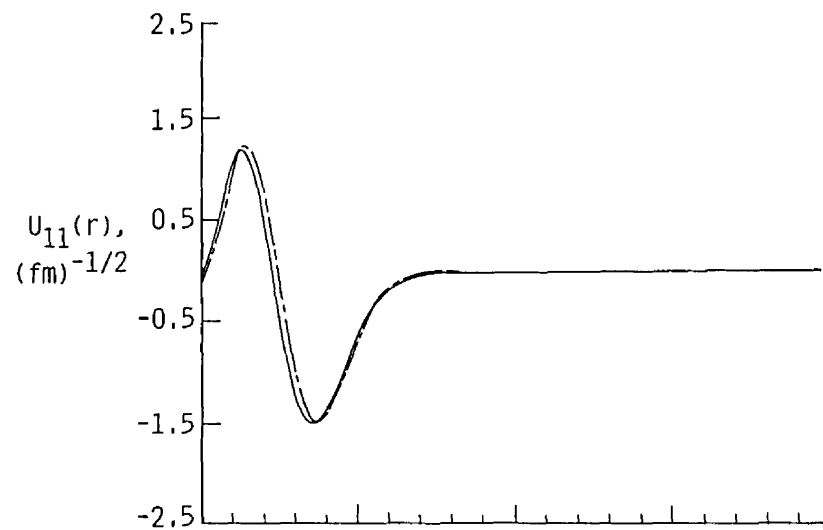
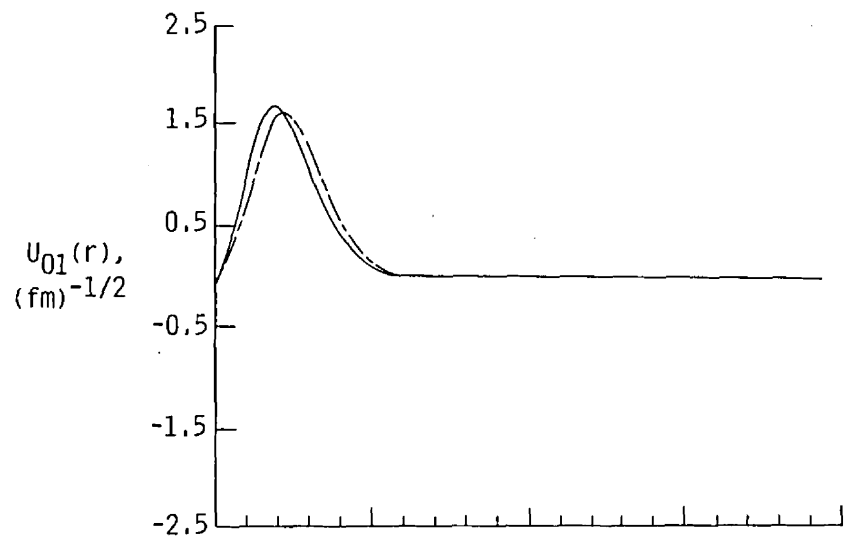


Figure 9. Bottom radial wave function $U_{n_r \ell}(r)$ for $n_r = 0, 1, 2$, and 3 and $\ell = 1$.

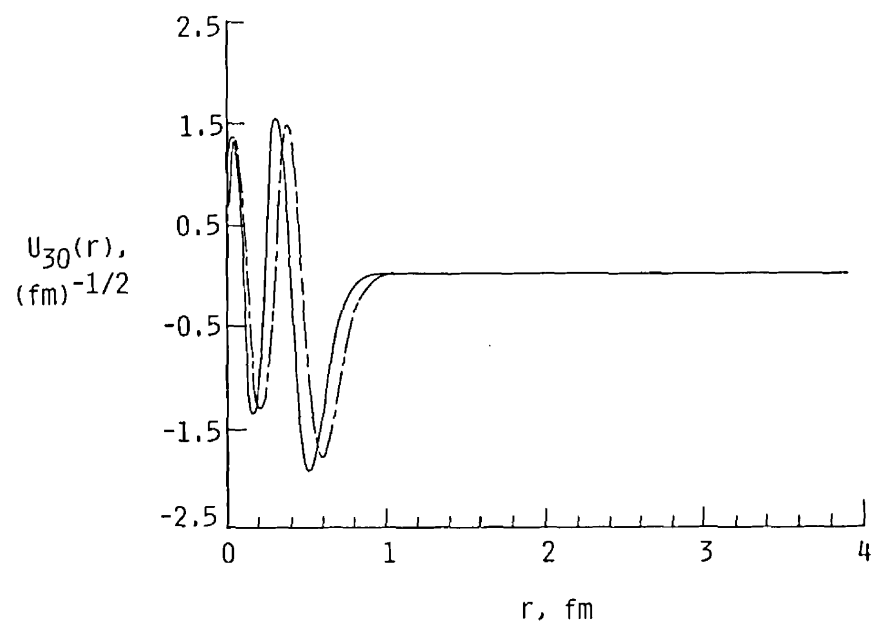
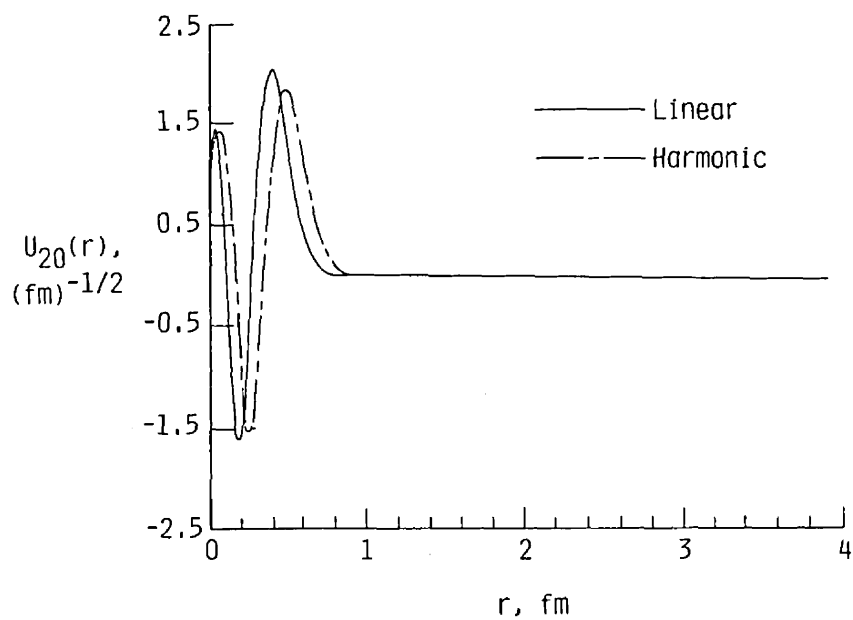
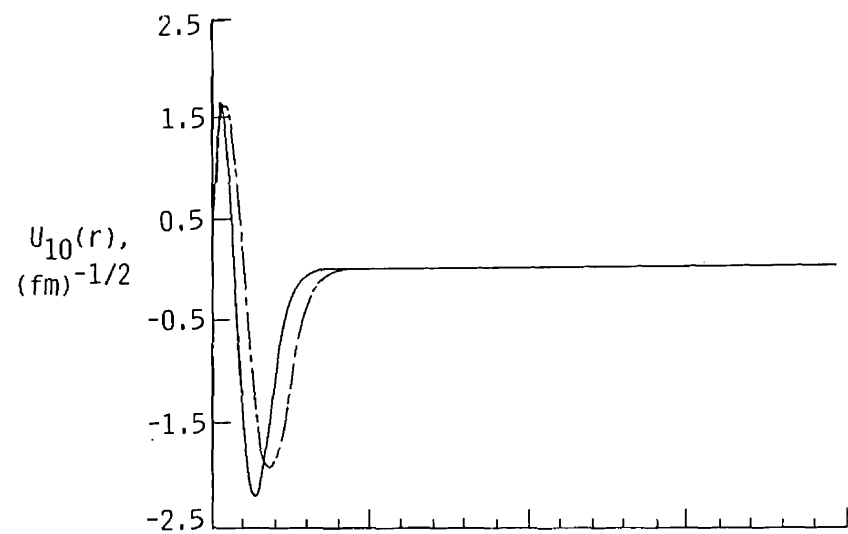
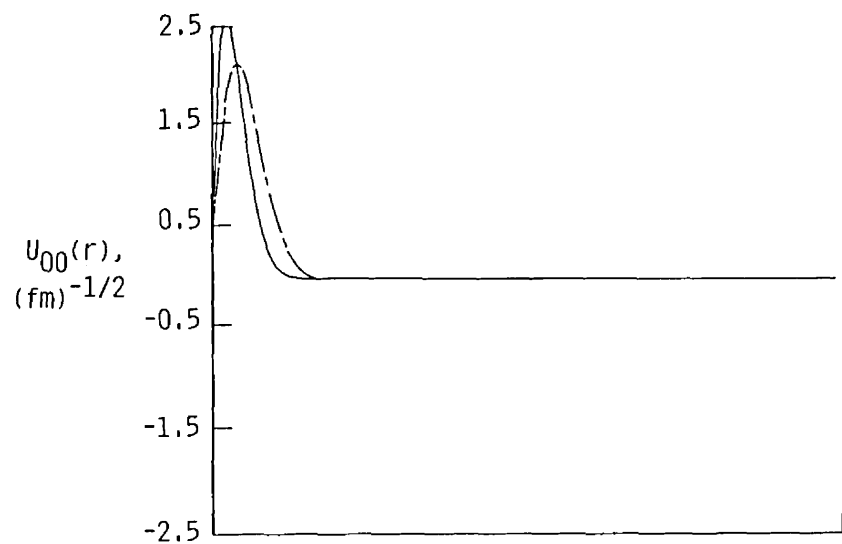


Figure 10. Top radial wave function $U_{n_r\ell}(r)$ for $n_r = 0, 1, 2$, and 3 and $\ell = 0$.

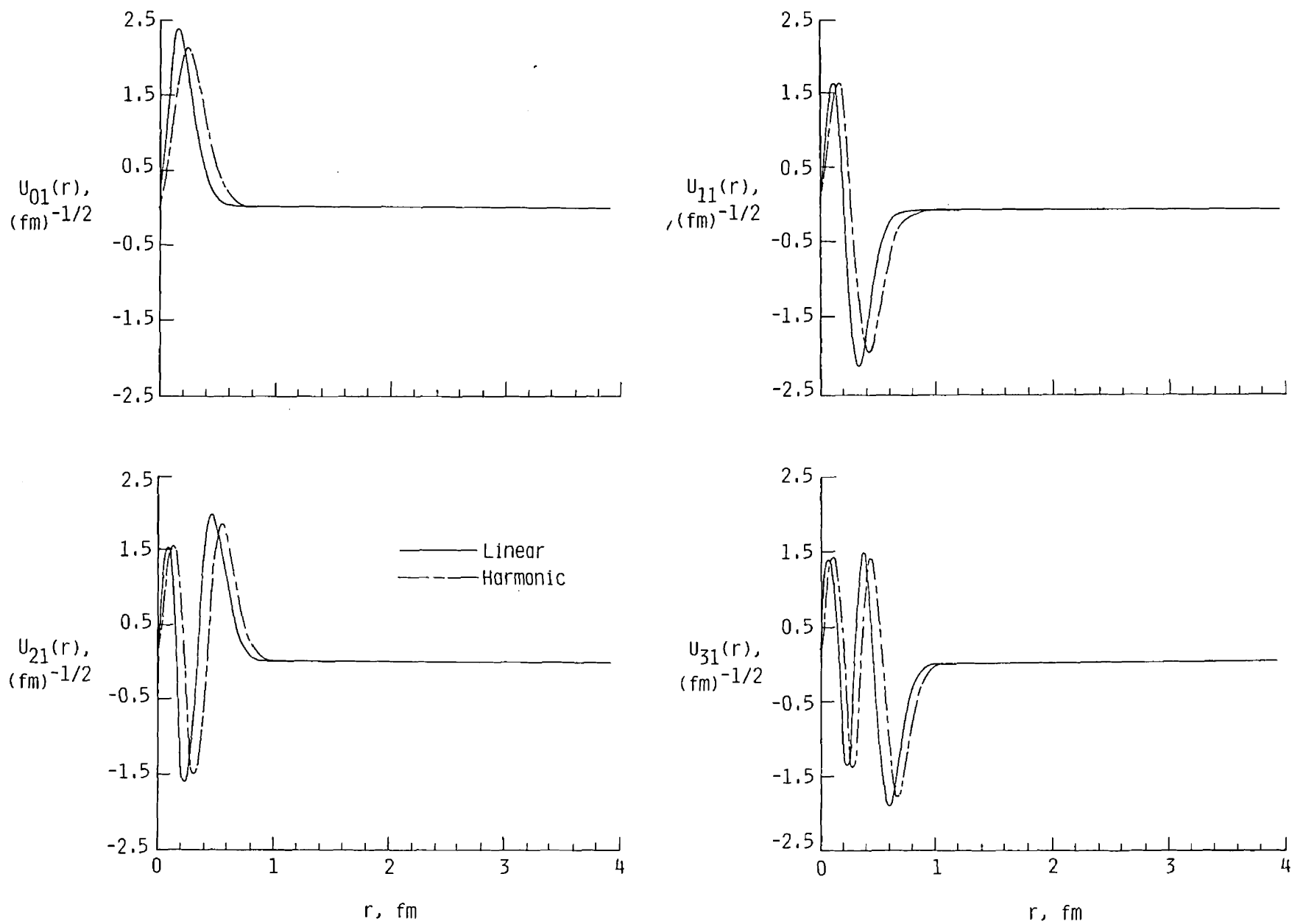


Figure 11. Top radial wave function $U_{n_r \ell}(r)$ for $n_r = 0, 1, 2$, and 3 and $\ell = 1$.

Report Documentation Page

1. Report No. NASA TM-4042		2. Government Accession No.		3. Recipient's Catalog No.	
4. Title and Subtitle Accuracy of Analytic Energy Level Formulas Applied to Hadronic Spectroscopy of Heavy Mesons				5. Report Date July 1988	
				6. Performing Organization Code	
7. Author(s) Forooz F. Badavi, John W. Norbury, John W. Wilson, and Lawrence W. Townsend				8. Performing Organization Report No. L-16434	
9. Performing Organization Name and Address NASA Langley Research Center Hampton, VA 23665-5225				10. Work Unit No. 199-22-76-01	
				11. Contract or Grant No.	
12. Sponsoring Agency Name and Address National Aeronautics and Space Administration Washington, DC 20546-0001				13. Type of Report and Period Covered Technical Memorandum	
				14. Sponsoring Agency Code	
15. Supplementary Notes Forooz F. Badavi: Planning Research Corporation, Hampton, Virginia. John W. Norbury: Washington State University, Pullman, Washington. John W. Wilson and Lawrence W. Townsend: Langley Research Center, Hampton, Virginia.					
16. Abstract In this work, linear and harmonic potential models are used in the nonrelativistic Schrödinger equation to obtain particle mass spectra for mesons as bound states of quarks. The main emphasis is on the linear potential where exact solutions of the S -state ($\ell = 0$) eigenvalues and eigenfunctions and the asymptotic solution for the higher order partial waves ($\ell > 0$) are obtained. A study of the accuracy of two analytical energy level formulas (Cornwall and Sheehy and Von Baeyer) as applied to heavy mesons is also included. Cornwall's formula is found to be particularly accurate and useful as a predictor of heavy quarkonium states. Exact solution for all partial waves of eigenvalues and eigenfunctions for a harmonic potential is also obtained and compared with the calculated discrete spectra of the linear potential. Detailed derivations of the eigenvalues and eigenfunctions of the linear and harmonic potentials are presented in appendixes.					
17. Key Words (Suggested by Authors(s)) Quarks Hadron spectroscopy Mesons				18. Distribution Statement Unclassified-Unlimited Subject Category 73	
19. Security Classif.(of this report) Unclassified		20. Security Classif.(of this page) Unclassified		21. No. of Pages 25	
				22. Price A02	

**National Aeronautics and
Space Administration
Code NTT-4**

**Washington, D.C.
20546-0001**

**Official Business
Penalty for Private Use, \$300**

**BULK RATE
POSTAGE & FEES PAID
NASA
Permit No. G-27**



**POSTMASTER: If Undeliverable (Section 158
Postal Manual) Do Not Return**
